RISK PROCESS WITH UNCERTAIN CLAIMS AMOUNT

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Abstract

The contribution is devoted to the risk process in which the claims amount are uncertain. The uncertainty is modeled using the randomness and fuzzines simultaneously and the claim amount are treated as the fuzzy random variable. The definition of the fuzzy random variables proposed by Kwakernaak (1978, 1979) is used in this contribution. The problem connected with the ruin of such process is investigated and some numerical examples are presented. The situation when the initial capital is equal zero is studied and the fuzzy ruin and the mean value of it are computed for the fuzzy exponential random variables. The spread of fuzzy ruin, the measure of uncertainty, is investigated.

Keywords*: risk process, probability of ruin, fuzzy numbers, fuzzy random variables, uncertainty*

JEL Codes*: G22, C6*

1. Introduction

In the classical risk models the amount of claims X_i and the interarrival times T_i were treated as the random variables. It is stochastic approach, in which X_i are independent and identically distributed random variables like variables T_i, which also generate the Poisson process. But we can generalize this model and assume that claim amount may also be imprecision and uncertain. For instance, the reimbursement may be uncertain and the insurance company can not forecast it precisely. It is an approximate amount and we can treat it as fuzzy number. Furthermore, the parameters of the considered distributions can be imprecisely defined. So, the variables X_i cannot be modelled only by using the stochastic methods. They contains both the randomness and fuzziness, which require to be investigated simultaneously. The introduction and examining the fuzzy random variables enables us to better analyze the risk processes. We can more accurately and more realistic determine the probability of ruin in this situation.

We treat claim amount X_i as the fuzzy random variables introduced by Kwakernaak (1978) and developed among others by Puri and Ralescu (1986), Shapiro (2012) and Andres-Sanchez (2017). Liu and Liu (2003) gave another definition of fuzzy random variables based on the fuzzy variables. But we use the concept of the fuzzy random variable, the expected value of it and the cumulative distribution function done by Puri and Ralescu (1986) based on the random sets generated by α-cuts of the fuzzy numbers. We also study the fuzzy random variables, which are the product of the random variable and the fuzzy number. The fuzzy exponential random variable is an example of such fuzzy random variable.

We investigate the ruin problem, the important topic in the risk theory. The probability of ruin is the fuzzy number in this case. We also use the mean value of such probability. It is the scalar, the crisp value. We give the value of probability of ruin for zero initial capital and we present the exact value of it if the claim amount is an exponentially distributed fuzzy random variables. We received similar results as in the case of classical risk theory. The similar investigations were done by Huang *et al*. (2009). But they used the another definition of the fuzzy random variable based on the fuzzy variables. Zhao and Tang (2006) investigated the fuzzy random renewal process, strictly connected with the risk process, generated by a sequence the fuzzy random interarrival times. The discrete-time insurance risk model with fuzzy return rate was studied by Vernic (2016).

We expand the issues raised at Huang *et al*. (2009) paper. We consider the case, when we only know the approximate value of the parameter of the investigated distribution of the claim amount. We investigate also the influence of the initial capital and the premium on the probability of ruin and on the spread, the measure of uncertainty.

The second section is devoted to the fuzzy random variables. We presented the basic notions and properties connected with the fuzzy numbers: the mean values and arithmetical operations on fuzzy numbers. We introduce the definition of a fuzzy random variable, the expected value and the cumulative distribution function of it. Also we study the sum of the fuzzy random variables. The third section present the risk model with uncertain claim amounts, mainly the ruin problem. First we recall the main information connected with the classical ruin theory. Next we present the version with the fuzzy claim amount. We give the exact solution for the fuzzy exponential random variables. Finally, we present the numerical examples.

2. Fuzzy random variables

The uncertainty is modeled by the fuzzy sets in our paper. The fuzzy set **A** defined on the space X is characterized by the its membership function $\mu_{\mathbf{A}}: X \rightarrow [0, 1]$ (see, Zadeh, 1965; Negoita and Ralescu, 1975; Dubois and Prade, 1980). The value $\mu_{\mathbf{A}}(x)$ represents the degree of membership of the element x to the fuzzy set **A**. The value 0 indicates non-membership in **A** and 1 indicates absolute membership. Every fuzzy set **A** can be represented by its α-cuts. They are the crisp sets $A_\alpha = \{x \in X \mid \mu_{\mathbf{A}}(x) \geq \alpha\}$ for any $0 < \alpha \leq 1$ and A_0 called the support of **A** is the closure of set $\{x \in X \mid \mu_{\mathbf{A}}(x) > 0\}$. The set A_1 is called the core of **A**. We will denote the fuzzy sets by the bold letters, e.g. **A**, and the crisp sets by the italic, non-bold letters, e.g. A_α.

Now we assume, that space X is a real line R and the support and the core of fuzzy set **A** are the compact intervals, i.e. $A_0 = [a, d]$ and $A_1 = [b, c]$, where $a, b, c, d \in$ R. In addition, suppose that the $\mu_{\mathbf{A}}$ is continuous function and it is strictly increasing on $[a, b]$ and strictly decreasing on $[c, d]$. So, every α-cut is the compact interval, i.e. $A_\alpha = [A_\alpha^L, A_\alpha^U]$. Such fuzzy sets are called the fuzzy numbers. The class of all fuzzy numbers is denoted as *FN*. The trapezoidal fuzzy numbers $\mathbf{A} = (a, b, c, d)$ has the linear membership function on the intervals $[a, b]$ and $[c, d]$. When $b = c$ we obtain the triangular fuzzy number $\mathbf{A} = (a, b, d)$. In practice, the construction of the trapezoidal fuzzy number can be done by the expert. For instance, he presented the elements definitely belonging to **A** and the elements definitely not belonging to it. The first set is the core and the complement of second is the support. The width of the support $d - a$ can be treated as the degree of imprecision and called a spread $sp(\mathbf{A})$.

The mean value of the fuzzy number **A** is defined by formula

$$M(\mathbf{A}) = \frac{1}{2}\int_0^1 (A_\alpha^L + A_\alpha^U)\mathrm{d}\alpha\,, \tag{1}$$

For the trapezoidal fuzzy number **A** we have

$$A_\alpha^L = (b-a)\alpha + a, \quad A_\alpha^U = (c-d)\alpha + d, \quad M(\mathbf{A}) = \frac{a+b+c+d}{4}. \tag{2}$$

We can define the arithmetic operations on fuzzy numbers based on the α-cuts of the fuzzy numbers. Let **A**, **B** $\in$ *FN* and $\lambda \in$ R, then

$$(A+B)_\alpha^L = A_\alpha^L + B_\alpha^L, \quad (A+B)_\alpha^U = A_\alpha^U + B_\alpha^U, \tag{3}$$

$$(A\cdot B)_\alpha^L = \min\{A_\alpha^L \cdot B_\alpha^L,\ A_\alpha^L \cdot B_\alpha^U,\ A_\alpha^U \cdot B_\alpha^L,\ A_\alpha^U \cdot B_\alpha^U\}, \tag{4}$$

$$(A\cdot B)_\alpha^U = \max\{A_\alpha^L \cdot B_\alpha^L,\ A_\alpha^L \cdot B_\alpha^U,\ A_\alpha^U \cdot B_\alpha^L,\ A_\alpha^U \cdot B_\alpha^U\}, \tag{5}$$

$$(\lambda A)_\alpha^L = \begin{cases} \lambda A_\alpha^L & \text{for } \lambda \ge 0 \\ \lambda A_\alpha^U & \text{for } \lambda < 0 \end{cases}, \quad (\lambda A)_\alpha^U = \begin{cases} \lambda A_\alpha^U & \text{for } \lambda \ge 0 \\ \lambda A_\alpha^L & \text{for } \lambda < 0 \end{cases}. \tag{6}$$

Fuzzy random variable **X** is a random variable whose outcomes are imprecision, uncertain values, describes as fuzzy numbers. Let (Ω, $\mathcal{A}$, P) be the probability space (see Kwakernaak, 1978, 1979; Puri and Ralescu, 1986), than

$$\mathbf{X}: \Omega \to FN. \tag{7}$$

So, for every $\omega \in \Omega$, $(\mathbf{X}(\omega))_\alpha = [X_\alpha^L(\omega), X_\alpha^U(\omega)]$ and we can treat the fuzzy random variable **X** as the family of the random variables X_α^L, X_α^U, where $0 \le \alpha \le 1$. We can define the expected value of **X** as a fuzzy number **E**(**X**) whose α-cuts take the form

$$(\mathbf{E}(\mathbf{X}))_\alpha = [E(X_\alpha^L), \mathrm{E}(X_\alpha^U)] \tag{8}$$

or as crisp number E(**X**) equals the mean of **E**(**X**), i.e.

$$E(\mathbf{X}) = \frac{1}{2}\int_0^1 (E(X_\alpha^L) + \mathrm{E}(X_\alpha^U))\mathrm{d}\alpha. \tag{9}$$

The probability of fuzzy event $\mathbf{F}(x) = P(\mathbf{X} \le x)$ (fuzzy cumulative distribution function) is a fuzzy number whose α-cuts are equal

$$(\mathbf{F}(x))_\alpha = [P(X_\alpha^U \le x),\ P(X_\alpha^L \le x)] = [F_\alpha^U(x),\ F_\alpha^L(x)]. \tag{10}$$

In this paper we will use the fuzzy random variable **X**, which takes the form **X** = Y**A**, where Y is a random variable and **A** is a fuzzy number (see, Huang *et al*., 2009). The α-cuts of it are equal $X_\alpha^L = A_\alpha^L Y$ and $X_\alpha^U = A_\alpha^U Y$. The expected value of **X** is equal **E**(**X**) = $E(Y)$**A** with mean value E(**X**) = $E(Y)M$(**A**).

Let Y be a exponentially distributed random variable, $E(Y) = m$, i.e. $Y \sim \mathcal{E}(m)$. So, X_α^L and X_α^U are the exponentially distributed random variables with parameters equal to $A_\alpha^L m$ and $A_\alpha^U m$, respectively. We will say that **X** is a exponentially distributed fuzzy random variable denoted **X** ~ $\mathcal{EF}(m$, **A**). The expected value of such fuzzy random variable is equal **E**(**X**) = m**A** with mean value E(**X**) = mM(**A**).

Example 1. Let **X** ~ $\mathcal{EF}$(2, **A**) and **A** = (1, 3, 4). Then $A_\alpha = [1+2\alpha,\ 4-\alpha]$, M(**A**) = 2.75, X_α^L ~ $\mathcal{E}$(2 + 4α) and X_α^U ~ $\mathcal{E}$(8 – 2α). The expected value of such exponentially distributed fuzzy random variable is a triangular fuzzy number too and **E**(**X**) = 2**A** = (2, 6, 8) with mean value E(**X**) = 5.5. The α-cuts of fuzzy cumulative distribution function **F**(x) are determined by the following cumulative distribution functions

$$F_\alpha^U(x) = 1 - \exp\left(-\tfrac{x}{2+4\alpha}\right), \qquad F_\alpha^L(x) = 1 - \exp\left(-\tfrac{x}{8-2\alpha}\right). \tag{11}$$

We see that for fixed value x, the **F**(x) is not the triangular fuzzy number. The mean value of it is equal

$$M(\mathbf{F}(x)) = \int_0^1 \left(1 - \tfrac{1}{2}\left(\exp\left(-\tfrac{x}{2+4\alpha}\right) + \exp\left(-\tfrac{x}{8-2\alpha}\right)\right)\right) \mathrm{d}\alpha\ . \tag{12}$$

This integral has not the evident form. We must use the numeric methods to compute it.

Let Y_i, where i = 1, 2, …, n be a sequence of idd. random variables, $E(Y_i) = m$ and let $\mathbf{X}_i = Y_i\mathbf{A}$. We will study the sum $\mathbf{S}_n$ of such fuzzy random variable:

$$\mathbf{S}_n = \sum_{i=1}^{n} \mathbf{X}_i = \left(\sum_{i=1}^{n} Y_i\right)\mathbf{A} = C_n\mathbf{A}. \tag{13}$$

The α-cuts of it take the form

$$\left(S_n\right)_\alpha = [A_\alpha^L C_n,\ A_\alpha^U C_n], \tag{14}$$

the expected value is equal $\mathbf{E}(\mathbf{S}_n) = nm\mathbf{E}(\mathbf{A})$ and $E(\mathbf{S}_n) = nmM(\mathbf{A})$. When Y_i are exponentially distributed random variables, i.e. Y_i ~ $\mathcal{E}(\lambda)$, then C_n has gamma distribution Ga(n, m) and $\mathbf{E}(\mathbf{S}_n) = nm\mathbf{E}(\mathbf{A})$.

These fuzzy random variables can have the following interpretation. If the core of the fuzzy number **A** contains 1, then **X**(ω) = $Y(\omega)$**A** can be treated as the approximate value (Heilpern, 1981) "about" $Y(\omega)$, e.g. the approximate value of the parameter of the exponentially distributed random variable.

Example 2. Let **A** = (0.9, 1, 1.2) and Y ~ $\mathcal{E}$(2). Then **X** = Y**A** can be interpret as the exponentially distributed random variables with the approximate values. The random variables X_α^L ~ $\mathcal{E}$(1.8 + 0.2α), X_α^U ~ $\mathcal{E}$(2.4 – 0.4α) and **E**(**X**) = (1.8, 2, 2.4) with the mean value E(**X**) = 2.05.

We may create the fuzzy random variables in another way. Let $\mathcal{L}$ be the family of the random variables indexed by the parameter m. We assume, that we know the approximate value **M** of the value of the parameter m only. So, we obtain the fuzzy subset of the random variables **X** of the family $\mathcal{L}$, with membership function $\mu_{\mathbf{X}}(X) = \mu_{\mathbf{M}}(m)$ for $X \sim \mathcal{L}(m)$. The ends of the α-cuts are the random variables: $X_\alpha^L \sim \mathcal{L}(M_\alpha^L)$ and $X_\alpha^U \sim \mathcal{L}(M_\alpha^U)$. For the family of the exponential distributed random variables and **M** = m**A**, we obtain $X_\alpha^L \sim \mathcal{E}(A_\alpha^L m)$ and $X_\alpha^U \sim \mathcal{E}(A_\alpha^U m)$. So, these are the same ends of α-cuts as for the fuzzy random variable Y**A**, where Y ~ $\mathcal{E}$(m). These two methods of the construction of the fuzzy random variables is the same in this case. But, for other distributions we can obtain different results.

Example 3. Let $\mathcal{L}$ be the family of Pareto distributed random variables X indexed by parameter a, i.e. $F_X(x) = 1 - \left(\frac{1}{x+1}\right)^a$ for $x \geq 0$, **M** = 2**A**, where **A** = (0.9, 1, 1.2) and a = 2. Then the 0.5-cuts of the fuzzy subset **X** of the family $\mathcal{L}$ induced by the approximate parameter **M** have the following cumulative distribution functions:

$$F_{X_{0.5}^L} = 1 - \left(\frac{1}{x+1}\right)^{1.9}, \qquad F_{X_{0.5}^U} = 1 - \left(\frac{1}{x+1}\right)^{2.2}. \tag{13}$$

They are the Pareto distributed random variables with the parameters 1.9 and 2.2, i.e. $X_{0.5}^L$ ~ Pa(1.9) and $X_{0.5}^U$ ~ Pa(2.2). Now, we investigate the fuzzy random variable **Z** = Y**A**, where Y ~ Pa(2). Then

$$F_{Z_{0.5}^L} = 1 - \left(\frac{1.9}{x+1.9}\right)^{2}, \qquad F_{Z_{0.5}^U} = 1 - \left(\frac{2.2}{x+2.2}\right)^{2}. \tag{14}$$

These 0.5-cuts are not Pareto distributed. So, we obtain the different results. In this case we obtain **E**(**Z**) = **M** = (1.8, 2, 2.4).

3. Risk model with uncertain claims amount

Let X_i, where i = 1, 2, …, be a sequence of idd. positive random variables and $E(X_i) = m$. Random variable X_i denotes the amount of i-th claim. The interarrival time between the (i – 1)th and i-th claim is described by the random variable T_i. We assume, that $E(T_i) = 1/\lambda$ are idd. exponentially distributed random variables. Let

$$N(t) = \max_{n \geq 0} \left\{ n \mid 0 < \sum_{i=1}^{n} T_i \leq t \right\}, \tag{15}$$

where $\sum_{i=1}^{0} T_i = 0$ and $N(t)$ is the number of claims by time t. So, it is a Poisson process with intensity λ. The classical risk model takes the form (e.g. Gerber, 1979; Rolski *et al*., 1999)

$$U(t) = u + ct - S(t), \tag{16}$$

where $U(t)$ is the insurer's capital at time t, $u \geq 0$ is initial capital, c is the insurer's premium income per unit time and $S(t) = \sum_{i=1}^{N(t)} X_i$ is the aggregate claims by time t.

The time of ruin T is denoted by formula

$$T = \inf\{t \mid U(t) < 0\}. \tag{17}$$

If $T = \infty$, then the ruin does not occur. We can investigate the probability of ruin

$$\psi(u) = P(T < \infty \mid U(0) = u) \tag{18}$$

treated as function of initial capital. When

$$c \leq \lambda m \tag{19}$$

the probability of ruin is equal 1 for every initial capital u. So, we assume that $c > \lambda m$. Also, for the grater initial capital u the probability of ruin is close to zero, i.e. $\lim_{u \to \infty} \psi(u) = 0$. For the initial capital u = 0 the probability of ruin is equal

$$\psi(0) = \tfrac{\lambda}{c} m. \tag{20}$$

When the claims are exponentially distributed random variables with parameter *m*, than the probability of ruin has the evident form:

$$\psi(u) = \tfrac{\lambda}{c} m \exp\left(-u\left(\tfrac{1}{m} - \tfrac{\lambda}{c}\right)\right). \tag{21}$$

Now assume, that the amounts of claims are imprecision, uncertain and random. We can treat they as the fuzzy random variable $\mathbf{X}_i$ and they take the form $\mathbf{X}_i = Y_i\mathbf{A}$ introduced in section 2. Also, the fuzzy number **A** satisfies condition, that $A_1^L \le 1 \le A_1^U$.

The aggregate claims $\mathbf{S}(t)$ by time *t* and the insurer's surplus $\mathbf{U}(u)$ are the FRV in this case. They are equal

$$\mathbf{S}(t) = \sum_{i=1}^{N(t)} \mathbf{X}_i = \left(\sum_{i=1}^{N(t)} Y_i\right)\mathbf{A} = D(t)\mathbf{A}. \tag{22}$$

$$\mathbf{U}(t) = u + ct - \sum_{i=1}^{N(t)} \mathbf{X}_i = u + ct - D(t)\mathbf{A} \tag{23}$$

The α-cuts of $\mathbf{U}(t)$ take the form

$$U(t)_\alpha^L = u + ct - D(t)A_\alpha^U, \qquad U(t)_\alpha^U = u + ct - D(t)A_\alpha^L. \tag{24}$$

Let $\psi(u)_\alpha^L$ be the probability of ruin for risk process $U(t)_\alpha^U$ and $\psi(u)_\alpha^U$ for $U(t)_\alpha^L$. We have $U(t)_\alpha^L \le U(t)_\alpha^U$, $U(t)_\alpha^L \le U(t)_\beta^L$ and $U(t)_\alpha^U \ge U(t)_\beta^U$ for $\alpha \le \beta$. Then the probabilities of ruin satisfy the similar inequalities: $\psi(u)_\alpha^L \le \psi(u)_\alpha^U$, $\psi(u)_\alpha^L \le \psi(u)_\beta^L$ and $\psi(u)_\alpha^U \ge \psi(u)_\beta^U$. So, we can treat $\psi(u)_\alpha^L$ and $\psi(u)_\alpha^U$ as the α-cuts of some fuzzy number for fixed value of initial capital *u*. We denote it as $\boldsymbol{\psi}(u)$ and interpreted it as fuzzy probability of ruin. Let $m_\alpha^L = mA_\alpha^L$ and $m_\alpha^U = mA_\alpha^U$. When if we want to calculate the probability of ruin, we must remember that $\psi(u)_\alpha^L = 1$ if $c \le \lambda m_\alpha^L$ and $\psi(u)_\alpha^U = 1$ if $c \le \lambda m_\alpha^U$. The right side of (19) is the trapezoidal fuzzy number $\mathbf{C} = \lambda m\mathbf{A} = (a_C, b_C, c_C, d_C)$ and we should consider the position of *c* relative to this fuzzy set.

Example 4. Assume that intensity $\lambda = 3$ and the expected value $E(Y_i) = m = 2$, $\mathbf{A} = (0.9, 1, 1.2)$.

a) Let the premium $c = 10$. Then α-cut $A_\alpha = [0.9 + 0.1\alpha, 1.2 - 0.2\alpha]$. We designate the probability of ruin when the initial capital $u = 0$. We have $U(t)_\alpha^L = 10t - (1.2 - 0.2\alpha)D(t)$ and $U(t)_\alpha^U = 10t - (0.9 + 0.1\alpha)D(t)$. Using (20) we obtain the α-cuts of the probability of ruin

$$\psi(0)_\alpha^L = 0.54 + 0.06\alpha, \qquad \psi(0)_\alpha^U = 0.72 - 0.12\alpha. \tag{25}$$

The fuzzy probability of ruin takes the form $\boldsymbol{\psi}(0) = \tfrac{\lambda m}{c}\mathbf{A} = 0.6\mathbf{A}$ in this case. It is triangular fuzzy number $\boldsymbol{\psi}(0) = (0.54, 0.6, 0.72)$ and its mean value is equal $M(\mathbf{A}) = 0.615$. The right side of (19) is equal $\mathbf{C} = (5.4, 6, 7.2)$ and $d_C < c$.

b) Let c = 6.2 and u = 0. Then, we have $\lambda m_\alpha^L = 5.4 + 0.6\alpha$, $\lambda m_\alpha^U = 7.2 - 1.2\alpha$. In this case the core of $\boldsymbol{\psi}(0)$ is equal {0.9677}, but $c < d_C$ and the fuzzy ruin $\boldsymbol{\psi}(0)$ is the following truncated fuzzy number:

$$\psi(0)_\alpha^L = 0.8710 + 0.0968\alpha, \quad \psi(0)_\alpha^U = \begin{cases} 1 & \text{for} \quad 0 \le \alpha \le 0.8336 \\ 1.1613 - 0.1935\alpha & \text{for} \quad 0.8336 < \alpha \le 1 \end{cases}.$$

$$\mu_{\boldsymbol{\psi}(0)}(p) = \begin{cases} 0 & \text{for} \quad 0 \le p \le 0.8710 \\ 10.3306p - 8.9979 & \text{for} \quad 0.8710 < p \le 0.9677 \\ -5.1680p + 6.0016 & \text{for} \quad 0.9677 < p \le 1 \end{cases} \tag{26}$$

The mean value of it is equal $M(\boldsymbol{\psi}(0))$ = 0.9584.

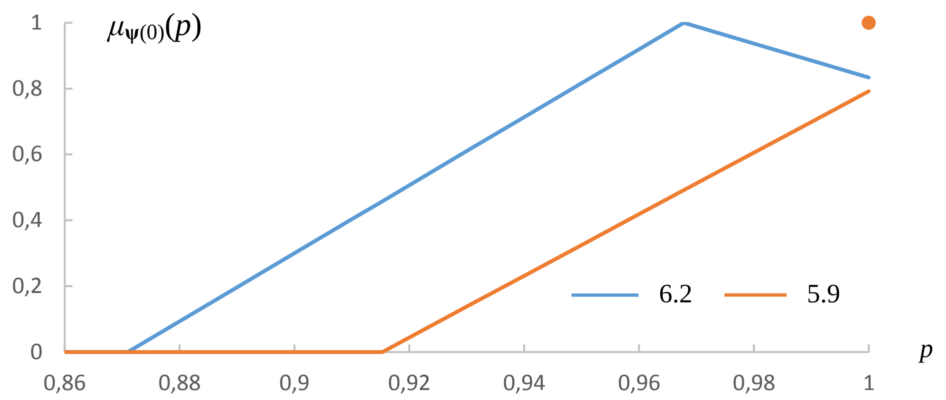
c) Let c = 5.9 and u = 0. Then $c < b_C$ and the fuzzy probability of ruin takes the form

$$\psi(0)_\alpha^L = \begin{cases} 0.9153 + 0.1017\alpha & \text{for} \quad 0 \le \alpha < 0.8333 \\ 1 & \text{for} \quad 0.8333 \le \alpha \le 1 \end{cases}, \qquad \psi(0)_\alpha^U = 1.$$

$$\mu_{\boldsymbol{\psi}(0)}(p) = \begin{cases} 0 & \text{for} \quad 0 \le p \le 0.9153 \\ 9.8333p - 9 & \text{for} \quad 0.9153 < p < 1 \\ 1 & \text{for} \quad p = 1 \end{cases} \tag{27}$$

It is the degenerate triangular fuzzy number $\boldsymbol{\psi}(0)$ = (0.9153, 1, 1) and $M(\boldsymbol{\psi}(0))$ = 0.9823. The graphs of the membership function of the fuzzy probability of ruin for the cases b) and c) is presented in Figure 1.

Figure 1. The graph of the membership function of the fuzzy probability of ruin for u = 0.



Source: Own elaboration

Now, we assume that the claim amounts are the exponentially distributed fuzzy random variables, i.e they take the form $\mathbf{X}_i = Y_i\mathbf{A}$, where $Y_i \sim \mathcal{E}(m)$ and $\mathbf{A}$ is a fuzzy number. Then α-cuts of $\mathbf{X}_i$ are equal

$$(X_i)_\alpha^L = A_\alpha^L Y_i, \qquad (X_i)_\alpha^U = A_\alpha^U Y_i. \tag{28}$$

They are exponentially distributed too with the parameters $A_\alpha^L m$ and $A_\alpha^U m$. So, we can give the evident form of the probability of ruin in this case (18). The α-cuts of them take the form

$$\psi(u)_\alpha^L = \tfrac{\lambda}{c} A_\alpha^L m \exp\left(-u\left(\frac{1}{A_\alpha^L m} - \frac{\lambda}{c}\right)\right), \qquad \psi(u)_\alpha^U = \tfrac{\lambda}{c} A_\alpha^U m \exp\left(-u\left(\frac{1}{A_\alpha^U m} - \frac{\lambda}{c}\right)\right). \tag{29}$$

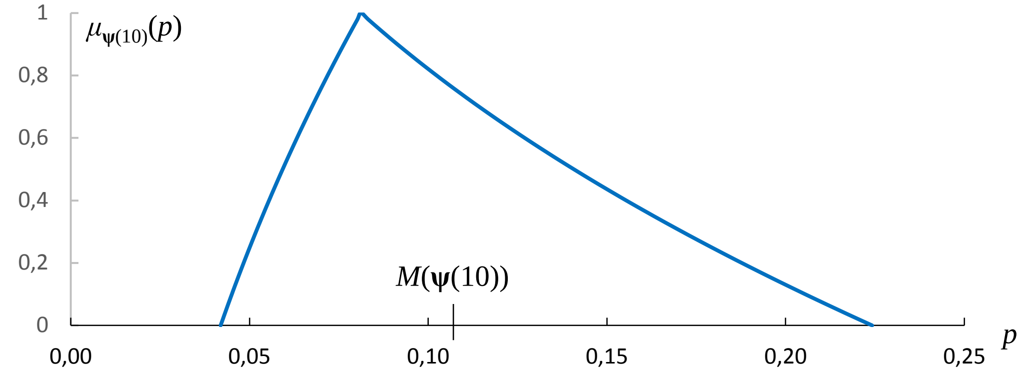
Example 5. Let $\lambda = 3$, $c = 10$ and the claims X_i are exponentially distributed. We do not know the exact value of the parameter m of such distribution, but we only received the information, that it is "about 2". If the approximate value **M** of parameter m takes the form $\mathbf{M} = 2\mathbf{A}$, where $\mathbf{A} = (0.9, 1, 1.2)$, then the α-cuts of the fuzzy probability of ruin are equal

$$\psi(u)_\alpha^L = (0.54 + 0.06\alpha)\exp\left(0.3u - \tfrac{u}{1.8+0.2\alpha}\right), \tag{30}$$

$$\psi(u)_\alpha^U = (0.72 - 0.12\alpha)\exp\left(0.3u - \tfrac{u}{2.4-0.4\alpha}\right). \tag{31}$$

For $u > 0$ these fuzzy sets are not the triangular fuzzy numbers. For instance, the graph of the membership function of the fuzzy probability of ruin for initial capital $u = 10$ is presented in Figure 2.

Figure 2. The graph of the membership function of the fuzzy probability of ruin for $u = 10$.

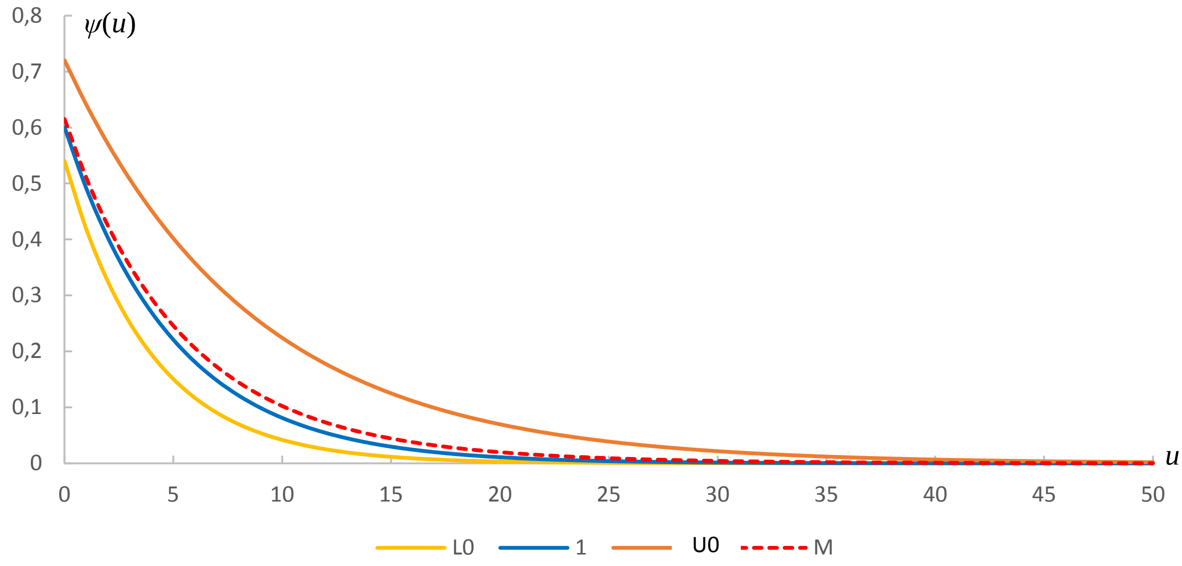


Source: Own elaboration

The sides of $\boldsymbol{\psi}(10)$ are not linear, then the mean value of this fuzzy number is the integral, which has not the evident form. We can use the Mathematica to obtain an approximate value of it. It is equal $M(\boldsymbol{\psi}(10)) = 0.1024$. The spread is equal 0.18 in this case.

The Figure 3 contain the graph of the values of the support: $\psi(u)_0^L$, $\psi(u)_0^U$, the core $\psi(u)_1^L$ and mean value of the fuzzy probability of ruin $\boldsymbol{\psi}(u)$ for different values of initial capital u. They are denoted as $L0$, $U0$, 1 and M respectively on this graph.

Figure 3. The graph of values of the support, the core and the mean of $\psi(u)$.



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We see that the probabilities of ruin decreases with growth of the initial capital u and tend to zero. They rapidly decreases when $u \leq 20$, from 0.615 to 0.020 (for $M(\psi(u))$, see Table 1).

Table 1. The values of the mean probability of ruin for different values u and c.

	c					
u	8	10	20	30	40	50
0	0.768750	0.615000	0.307500	0.205000	0.153750	0.123000
1	0.687331	0.510133	0.219538	0.139221	0.101837	0.080257
2	0.615588	0.423873	0.157007	0.094710	0.067568	0.052457
3	0.552281	0.352804	0.112479	0.064541	0.044908	0.034346
4	0.496334	0.294155	0.080718	0.044057	0.029899	0.022526
6	0.402928	0.205534	0.041782	0.020635	0.013321	0.009739
8	0.329316	0.144586	0.021774	0.009730	0.005975	0.004239
10	0.270937	0.102385	0.011423	0.004619	0.002698	0.001858
15	0.170991	0.044410	0.002340	0.000737	0.000380	0.000243
20	0.111861	0.019968	0.000497	0.000122	0.000056	0.000033
30	0.052259	0.004406	0.000025	0.000004	0.000001	0.000001
40	0.026581	0.001059	0.000001	0.000000	0.000000	0.000000
50	0.014294	0.000269	0.000000	0.000000	0.000000	0.000000

Source: own elaboration

Now, we study the influence the value of the premium c on the fuzzy probability of ruin $\psi(u)$ for the different values of the initial capital u. Table 1 contains the value of the mean probability of ruin in this case.

The mean probabilities of ruin decreases with growth of the premium c and tend to zero. For instance, if the initial capital $u = 10$, then they decreases from 0.271 for $c = 8$ to 0.002 for $c = 50$ (see Table 1).

Table 2 contains the values of spread *sp* for the different values of the initial capital *u* and the premium *c*. The last column presents the values of the relative spread, i.e. $sp(\mathbf{A})/M(\boldsymbol{\psi}(u))$. These values depend to a small extent on the premium *c*. The differences are observed only on the fifth place after the dot. Of course, the spread decreased with growth *u* and *c*. But the relative spread increases when the initial capital *u* increases, too. We observe for $u > 4$, that the spread is greater than the mean value of the probability of ruin.

Table 2. The values of the spread for different values *u* and *c*.

				c			
u	8	10	20	30	40	50	relative
0	0.225000	0.180000	0.090000	0.060000	0.045000	0.036000	0.2927
1	0.299776	0.222492	0.095750	0.060720	0.044416	0.035004	0.4361
2	0.357631	0.246253	0.091214	0.055023	0.039254	0.030475	0.5810
3	0.401547	0.256514	0.081780	0.046926	0.032651	0.024972	0.7271
4	0.434005	0.257215	0.070581	0.038525	0.026144	0.019697	0.8744
6	0.472457	0.241001	0.048992	0.024196	0.015619	0.011420	1.1726
8	0.485661	0.213229	0.032112	0.014350	0.008812	0.006252	1.4748
10	0.482358	0.182280	0.020336	0.008223	0.004803	0.003307	1.7803
15	0.436748	0.113433	0.005978	0.001882	0.000970	0.000620	2.5542
20	0.372899	0.066564	0.001657	0.000406	0.000185	0.000110	3.3336
30	0.254856	0.021489	0.000119	0.000018	0.000006	0.000003	4.8768
40	0.169495	0.006751	0.000008	0.000001	0.000000	0.000000	6.3765
50	0.111982	0.002107	0.000001	0.000000	0.000000	0.000000	7.8344

Source: own elaboration

4. Conclusion

The risk process with the uncertain, imprecisely defined claim amount was studied in the paper. We treated the values of such claim amount as the fuzzy number. We investigated the probability of ruin and we calculated of it when the claims amount had the fuzzy exponential distribution. We used simultaneously both random and fuzzy methods to solve such problem. The some numerical examples were presented. We showed that the probability of ruin decreased when the initial capital and the premium increased. We also observed that spread decreased in this case, but the relative spread increased.

We investigated risk model with continuous time. The further research will be extended on studying situation when the time is discrete. We also want to study the problem when the claim amount and the interarrival times are uncertain and they are modelled by fuzzy numbers.

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